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The $(2+1)$-dimensional sine–Gordon equation; integrability and localized solutions

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Abstract. In this paper, the $(2+1)$-dimensional sine–Gordon equation (2DSG) introduced by Konopelchenko and Rogers is investigated and is shown to satisfy the Painlevé property. A variable coefficient Hirota bilinear form is constructed by judiciously using the Painlevé analysis with a non-conventional choice of the vacuum solutions. First the line kinks are constructed. Then, exact localized coherent structures in the 2DSGI equation are generated by the collision of two non-parallel ghost solitons, which drive the two non-trivial boundaries present in the system. Also the reason for the difficulty in identifying localized solutions in the 2DSGII equation is indicated. We also highlight the significance of the asymptotic values of the boundaries of the system.

1. Introduction

Considerable effort has been given recently to generalize $(1+1)$-dimensional soliton equations to $(2+1)$ dimensions (see, for example, [1–3]). Of these equations, the symmetric generalizations have gained considerable attention in the last decade, particularly after the identification of localized, exponentially decaying solutions [4, 5]. Notable amongst these equations are the Nizhnik–Novikov–Veselov (NNV) equation [6–8] and the Davey–Stewartson (DS) equation [4, 5, 9]. They represent, in turn, $(2+1)$-dimensional generalizations of the Korteweg–deVries equation and nonlinear Schrödinger equation, respectively, wherein the two spatial variables occur on an equal footing. It is natural to look for such types of generalization for the other ubiquitous equation, namely the sine–Gordon equation. Konopelchenko and Rogers [10, 11] have proposed an interesting symmetric generalization of the sine–Gordon equation to $(2+1)$ dimensions through a reinterpretation and generalization of a class of infinitesimal Bäcklund transformations originally introduced in gas dynamics by Loewner [12] as far back as in 1952 to give the system of equations

$$\left[\frac{\phi_x}{\sin\theta}\right]_x - \left[\frac{\phi_y}{\sin\theta}\right]_y + \frac{(\phi_y\theta_x - \phi_x\theta_y)}{\sin^2\theta} = 0 \qquad (1a)$$

$$\left[\frac{\phi'_x}{\sin\theta}\right]_x - \left[\frac{\phi'_y}{\sin\theta}\right]_y + \frac{(\phi'_x\theta_y - \phi'_y\theta_x)}{\sin^2\theta} = 0 \qquad (1b)$$

where $\theta_t = \phi + \phi'$. If we assume that $\phi' = 0$ and that $\theta_t = \phi$ is independent of y, then (1b) becomes trivial and (1a) gives the sine–Gordon equation

$$\theta_{xt} = \sin\theta\,. \qquad (2)$$

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Equation (1) has a number of equivalent representations and its localized solutions have been constructed by Dubrovsky and Konopelchenko [13] by using the $\bar{\partial}$ method. Even though the $(2+1)$-dimensional sine–Gordon (2DSG) equation is known to be completely integrable, its Painlevé property has not yet been established. In this paper, we address ourselves to this problem and carry out the singularity structure analysis by concentrating on a convenient form of the sine–Gordon equation and confirm its Painlevé nature. We also deduce its bilinear form straightforwardly from the Painlevé analysis using non-conventional vacuum solutions and construct exponentially localized structures using the Hirota method by driving the two boundaries through two non-parallel ghost solitons.

The plan of the paper is as follows. In section 2, we discuss the linearization and equivalent representation of the sine–Gordon equation apart from discussing its properties. In section 3, we carry out its singularity structure analysis and confirm its Painlevé property. Section 4 is concerned with the bilinearization and generation of line kinks of the 2DSG equation. Localized coherent solutions of 2DSGI equation are constructed in section 5 and the absence of such solutions for the 2DSGII equation is also discussed. Section 6 contains a short discussion of the results.

2. Linearization and equivalent representation

Equation (1) is known to arise as the compatibility condition [2, 10, 11] for the triad of operators $L_1 = \partial_x - S\partial_y$, $L_2 = \partial_t\partial_y - V\partial_y - W_y$, $L_3 = \partial_t\partial_x - V\partial_x - W_x$ where

$$S = -\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \qquad V = \frac{1}{2}\begin{bmatrix} 0 & -\theta_t \\ \theta_t & 0 \end{bmatrix}$$
$$W_y = -\frac{1}{2\sin\theta}\begin{bmatrix} \phi_x - \phi_y\cos\theta & \phi'_y\sin\theta \\ -\phi_y\sin\theta & -(\phi'_x + \phi'_y\cos\theta) \end{bmatrix} \tag{3}$$
$$W_x = -\frac{1}{2\sin\theta}\begin{bmatrix} \phi_y - \phi_x\cos\theta & \phi'_x\sin\theta \\ -\phi_x\sin\theta & -(\phi'_y + \phi'_x\cos\theta) \end{bmatrix}.$$

The commutators of the operators L_1, L_2 and L_3 are $[L_1, L_2] = S_y L_2$, $[L_1, L_3] = S_x L_3$, $[L_2, L_3] = 0$, when (1) is satisfied.

To analyse the 2DSG equation, it is convenient to look for a more elegant representation since equation (1) is rather complicated for further investigation. In fact, it has been shown [14] that one can indeed arrive at a more interesting representation for the 2DSG equation as the compatibility condition of a Lax-pair which is gauge equivalent to a pair constructed out of L_1, L_2, L_3 as

$$\phi_{\xi\eta t} + \tfrac{1}{2}\theta_\eta\rho_\xi + \tfrac{1}{2}\theta_\xi\rho_\eta = 0 \tag{4a}$$
$$\rho_{\xi\eta} = \tfrac{1}{2}(\theta_\xi\theta_\eta)_t \tag{4b}$$

where

$$\rho_\xi = -\frac{([\phi_\eta - \phi'_\eta] + \theta_{\eta t}\cos\theta)}{\sin\theta} \qquad \rho_\eta = \frac{([\phi_\xi - \phi'_\xi] - \theta_{\xi t}\cos\theta)}{\sin\theta} \tag{4c}$$

with the characteristic variables ξ and η having the form

$$\xi = \tfrac{1}{2}(y - \sigma x) \qquad \eta = \tfrac{1}{2}(y + \sigma x) \tag{5}$$

where $\sigma^2 = \pm 1$ and ρ is some potential. Here $\sigma^2 = 1$ corresponds to the sine–Gordon I equation and $\sigma^2 = -1$ to the sine–Gordon II equation. Eliminating ρ from (4), one obtains the single equation

$$\theta_{\xi\eta t} + m_1(\eta, t)\theta_\xi + m_2(\xi, t)\theta_\eta + \tfrac{1}{4}\theta_\eta\int_{-\infty}^{\eta}(\theta_\xi\theta_{\eta'})_t\,\mathrm{d}\eta' + \tfrac{1}{4}\theta_\xi\int_{-\infty}^{\xi}(\theta_\eta\theta_{\xi'})_t\,\mathrm{d}\xi' = 0 \tag{6}$$

where

$$m_1(\eta, t) = \lim_{\xi \to -\infty} \tfrac{1}{2}\rho_\eta(\xi, \eta, t) \tag{7a}$$

$$m_2(\xi, t) = \lim_{\eta \to -\infty} \tfrac{1}{2}\rho_\xi(\xi, \eta, t)\,. \tag{7b}$$

The usual $(1+1)$-dimensional sine–Gordon equation can be retrieved when the boundaries tend to a constant as

$$\theta_{\xi t} = -(m_1 + m_2)\sin\theta = -m\sin\theta \tag{8}$$

where the constant boundaries are the $(2+1)$-dimensional analogue of mass m in the $(1+1)$-dimensional sine–Gordon equation.

3. Singularity structure analysis

Before we carry out the singularity structure analysis [15], we effect the transformation

$$q = -\frac{\mathrm{i}\sigma}{2}\theta_\eta \qquad r = -\frac{\mathrm{i}\sigma}{2}\theta_\xi \tag{9}$$

in equation (4) and convert it into a system of three coupled equations as

$$q_{\xi t} + \tfrac{1}{2}\rho_\xi q + \tfrac{1}{2}\rho_\eta r = 0 \tag{10a}$$

$$r_{\eta t} + \tfrac{1}{2}\rho_\xi q + \tfrac{1}{2}\rho_\eta r = 0 \tag{10b}$$

$$\sigma^2\rho_{\xi\eta} = -2(qr)_t\,. \tag{10c}$$

We now consider a local Laurent expansion in the neighbourhood of a non-characteristic singular manifold $\phi(\xi, \eta, t) = 0$, $(\phi_t, \phi_\eta \neq 0)$. Assuming the leading orders of the solutions of equation (10) to have the form

$$q = q_0\phi^\alpha \qquad r = r_0\phi^\beta \qquad \rho = \rho_0\phi^\gamma \tag{11}$$

where q_0, r_0 and ρ_0 are analytic functions of (ξ, η, t), one can isolate the allowed values of α, β and γ. Substituting (11) in (10) and balancing the most dominant terms, we obtain

$$\alpha = \beta = \gamma = -1 \tag{12}$$

with

$$\rho_0 = 2\phi_t \qquad q_0^2 = \sigma^2\phi_\eta^2 \qquad r_0^2 = \sigma^2\phi_\xi^2\,. \tag{13}$$

To find the resonances, we now substitute the Laurent expansion of the solutions

$$\begin{aligned} q &= q_0\phi^{-1} + \cdots + q_j\phi^{j-1} + \cdots \\ r &= r_0\phi^{-1} + \cdots + r_j\phi^{j-1} + \cdots \\ \rho &= \rho_0\phi^{-1} + \cdots + \rho_j\phi^{j-1} + \cdots \end{aligned} \tag{14}$$

into equation (10) and equate the coefficients of ϕ^{j-3} to zero to give

$$\begin{bmatrix} (j^2 - 3j + 1)\phi_\xi\phi_t & -\phi_\eta\phi_t & \frac{(j-1)}{2}[q_0\phi_\xi + r_0\phi_\eta] \\ -\phi_\xi\phi_t & (j^2 - 3j + 1)\phi_\eta\phi_t & \frac{(j-1)}{2}[q_0\phi_\xi + r_0\phi_\eta] \\ 2r_0\phi_t(j-2) & 2q_0\phi_t(j-2) & \sigma^2(j-1)(j-2)\phi_\xi\phi_\eta \end{bmatrix} \begin{bmatrix} q_j \\ r_j \\ \rho_j \end{bmatrix} = 0\,. \tag{15}$$

For non-trivial solutions to exist, we require the resonance values to be

$$j = -1, 1, 1, 2, 2, 4\,. \tag{16}$$

The resonance at $j = -1$ represents the arbitrariness of the singular manifold $\phi(\xi, \eta, t) = 0$. In order to prove the existence of arbitrary functions at the other resonance values, we now substitute the full Laurent series (14) into (10). Collecting the coefficients of $(\phi^{-3}, \phi^{-3}, \phi^{-3})$, we end up with the equations (13). Now gathering the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-2})$, we obtain

$$-q_{0\xi}\phi_t - q_{0t}\phi_\xi - q_0\phi_{\xi t} + \tfrac{1}{2}[\rho_{0\xi}q_0 + \rho_{0\eta}r_0] - \tfrac{1}{2}[q_1\phi_\xi + r_1\phi_\eta]\rho_0 = 0 \tag{17a}$$

$$-r_{0\eta}\phi_t - r_{0t}\phi_\eta - r_0\phi_{\eta t} + \tfrac{1}{2}[\rho_{0\xi}q_0 + \rho_{0\eta}r_0] - \tfrac{1}{2}[q_1\phi_\xi + r_1\phi_\eta]\rho_0 = 0 \tag{17b}$$

$$\sigma^2[\rho_{0\xi}\phi_\eta + \rho_{0\eta}\phi_\xi + \rho_0\phi_{\xi\eta}] = -2[q_0r_0]_t + 2\phi_t[q_0r_1 + r_0q_1]\,. \tag{17c}$$

Substituting (13) in (17), it can be easily shown that the above three equations degenerate into a single equation

$$q_0r_1 + r_0q_1 + \sigma^2\phi_{\xi\eta} = 0\,. \tag{18}$$

The above equation implies that there is only one equation for the three variables q_1, r_1 and ρ_1 and hence two of them must be arbitrary.

Now, collecting the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-1})$, we have

$$q_{0\xi t} + \tfrac{1}{2}[\rho_{0\xi}q_1 - \rho_0\phi_\xi q_2 + \rho_{1\xi}q_0 + \rho_2\phi_\xi q_0 + \rho_{0\eta}r_1 - \rho_0\phi_\eta r_2 + \rho_{1\eta}r_0 + \rho_2\phi_\eta r_0] = 0 \tag{19a}$$

$$r_{0\eta t} + \tfrac{1}{2}[\rho_{0\xi}q_1 - \rho_0\phi_\xi q_2 + \rho_{1\xi}q_0 + \rho_2\phi_\xi q_0 + \rho_{0\eta}r_1 - \rho_0\phi_\eta r_2 + \rho_{1\eta}r_0 + \rho_2\phi_\eta r_0] = 0 \tag{19b}$$

$$\sigma^2\rho_{0\xi\eta} = -2(q_0r_1 + r_0q_1)_t\,. \tag{19c}$$

Using equations (17) and (18), one can easily check that (19*c*) is identically satisfied for both $\sigma^2 = \pm 1$ and that (19*a*) and (19*b*) reduce to the identical equation (since $q_{0\xi t} = r_{0\eta t}$)

$$q_2\phi_\xi\phi_t + r_2\phi_\eta\phi_t + \tfrac{1}{2}\rho_2(q_0\phi_\xi + r_0\phi_\eta) = q_{0\xi t} + q_1\phi_{\xi t} + r_1\phi_{\eta t} + \tfrac{1}{2}[\rho_{1\xi}q_0 + \rho_{1\eta}r_0]\,. \tag{20}$$

Thus, again we have only a single equation for three variables q_2, r_2 and ρ_2 and this suggests that two of them must be arbitrary corresponding to $j = 2, 2$. Next, collecting the coefficients of (ϕ^0, ϕ^0, ϕ^0) and solving the resultant equations, the set of functions (q_3, r_3, ρ_3) can be uniquely determined.

Now, from the coefficients of (ϕ^1, ϕ^1, ϕ^1), we obtain

$$5q_4\phi_\xi\phi_t - r_4\phi_\eta\phi_t + \tfrac{3}{2}\rho_4(q_0\phi_\xi + r_0\phi_\eta) = A \tag{21a}$$

$$5r_4\phi_\eta\phi_t - q_4\phi_\xi\phi_t + \tfrac{3}{2}\rho_4(q_0\phi_\xi + r_0\phi_\eta) = B \tag{21b}$$

$$4r_0\phi_tq_4 + 4q_0\phi_tr_4 + 6\sigma^2\phi_\xi\phi_\eta\rho_4 = C \tag{21c}$$

where

$$A = -q_{2\xi t} - 2q_{3\xi}\phi_t - 2q_{3t}\phi_\xi - 2q_3\phi_{\xi t} - \tfrac{1}{2}[\rho_{0\xi}q_3 + \rho_{1\xi}q_2 + \rho_2\phi_\xi q_2 + \rho_{2\xi}q_1] - \rho_3\phi_\xi q_1 \\ -\rho_3\phi_\eta r_1 - \tfrac{1}{2}[\rho_{3\xi}q_0 + \rho_{0\eta}r_3 + \rho_{1\eta}r_2 + \rho_2\phi_\eta r_2 + \rho_{2\eta}r_1 + \rho_{3\eta}r_0] \tag{22a}$$

$$B = -r_{2\eta t} - 2r_{3\eta}\phi_t - 2r_{3t}\phi_\eta - 2r_3\phi_{\eta t} - \tfrac{1}{2}[\rho_{0\xi}q_3 + \rho_{1\xi}q_2 + \rho_2\phi_\xi q_2 + \rho_{2\xi}q_1] - \rho_3\phi_\xi q_1 \\ -\rho_3\phi_\eta r_1 - \tfrac{1}{2}[\rho_{3\xi}q_0 + \rho_{0\eta}r_3 + \rho_{1\eta}r_2 + \rho_2\phi_\eta r_2 + \rho_{2\eta}r_1 + \rho_{3\eta}r_0] \tag{22b}$$

$$C = -2q_{0t}r_3 - 2q_{1t}r_2 - 2q_2\phi_tr_2 - 2q_{2t}r_1 - 4q_3\phi_tr_1 - 2q_{3t}r_0 - 2r_{0t}q_3 - 2r_{1t}q_2 - 2r_2\phi_tq_2 \\ -2r_{2t}q_1 - 4r_3\phi_tq_1 - 2r_{3t}q_0 - \sigma^2[\rho_{2\xi\eta} + 2\rho_{3\xi}\phi_\eta + 2\rho_{3\eta}\phi_\xi + 2\rho_3\phi_{\xi\eta}]\,. \tag{22c}$$

Analysing the above set of equations (21*a*)–(21*c*), they easily can be reduced to a set of two equations in three variables (q_4, r_4 and ρ_4) for both $\sigma^2 = \pm 1$ and hence one of them must be arbitrary corresponding to the resonance value $j = 4$. Thus, the general solution $\{q, r, \rho\}(\xi, \eta, t)$ of equation (10) admits the required number of arbitrary functions without the introduction of any movable critical manifold, thereby satisfying the Painlevé property. Thus, both the $(2 + 1)$-dimensional sine–Gordon I and II equations (4) are expected to be integrable.

4. Bilinearization and line solitons (line kinks)

Having proved the Painlevé nature of the (2+1)-dimensional sine–Gordon equation, we now proceed to obtain the other integrability properties like Bäcklund-transformation, bilinear form, line solitons and localized solutions (if admissible). To construct the Bäcklund transformation, we now truncate the Laurent series at the constant level term, that is $q_j = r_j = \rho_j = 0$ for $j \geqslant 2$ to yield

$$q = q_0\phi^{-1} + q_1 \qquad r = r_0\phi^{-1} + r_1 \qquad \rho = \rho_0\phi^{-1} + \rho_1 \tag{23}$$

where the pair of variables (q, q_1), (r, r_1) and (ρ, ρ_1) satisfy equation (10) while q_0, r_0 and ρ_0 are given by (13). The above equation (23) may be considered as an auto-Bäcklund transformation in the sense that we can use the vacuum solution to bilinearize the given nonlinear evolution equation (NLEE) to generate higher soliton solutions. Thus, the Hirota bilinear form can be constructed by considering the vacuum solution

$$q_1 = r_1 = 0 \qquad \rho_1 = 2\int_{-\infty}^{\xi} m_2(\xi', t)\,\mathrm{d}\xi' + 2\int_{-\infty}^{\eta} m_1(\eta', t)\,\mathrm{d}\eta' \tag{24}$$

where we have made use of the arbitrariness of ρ_1 to construct its vacuum solution. Here m_1 and m_2 are arbitrary functions of (η, t) and (ξ, t), respectively (cf equation (10)). With the above vacuum solution, the auto-Bäcklund transformation becomes

$$q = q_0/\phi = g/\phi \tag{25a}$$

$$r = r_0/\phi = h/\phi \tag{25b}$$

$$\rho = \rho_0/\phi + \rho_1 = 2\partial_t \log\phi + 2\int_{-\infty}^{\xi} m_2(\xi', t)\,\mathrm{d}\xi' + 2\int_{-\infty}^{\eta} m_1(\eta', t)\,\mathrm{d}\eta'\,. \tag{25c}$$

Equation (25) can be interpreted as the dependent variable transformation which helps in the bilinearization of equation (10). Using this, the Hirota bilinear form of equation (10) becomes

$$\mathrm{D}_\xi\mathrm{D}_t g\cdot\phi + m_2(\xi, t) g\cdot\phi + m_1(\eta, t) h\cdot\phi = 0 \tag{26a}$$

$$\mathrm{D}_\eta\mathrm{D}_t h\cdot\phi + m_2(\xi, t) g\cdot\phi + m_1(\eta, t) h\cdot\phi = 0 \tag{26b}$$

$$\sigma^2\mathrm{D}_\xi\mathrm{D}_\eta\phi\cdot\phi = -2gh \tag{26c}$$

$$h\mathrm{D}_\eta\mathrm{D}_t\phi\cdot\phi - g\mathrm{D}_\xi\mathrm{D}_t\phi\cdot\phi = 0 \tag{26d}$$

where D is the formal Hirota operator. An obvious interesting feature of the system (26) is the presence of coefficients $m_2(\xi, t)$ and $m_1(\eta, t)$ which are essential for the formation of dromion-like coherent structures as we point out below. The three dependent variables g, h and ϕ can be uniquely determined using equations (26a)–(26c) consistent with (26d).

To generate line kinks, we now expand the functions g, h and ϕ in the form of a power series in a small parameter ε as

$$\begin{aligned} g &= \varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots \\ h &= \varepsilon h^{(1)} + \varepsilon^3 h^{(3)} + \cdots \\ \phi &= 1 + \varepsilon^2\phi^{(2)} + \varepsilon^4\phi^{(4)} + \cdots. \end{aligned} \tag{27}$$

Substituting (27) in (26) and comparing the coefficients of various powers of ε, we obtain the following sets of equations:

ε:

$$g^{(1)}_{\xi t} + m_2(\xi, t)g^{(1)} + m_1(\eta, t)h^{(1)} = 0$$
$$h^{(1)}_{\eta t} + m_2(\xi, t)g^{(1)} + m_1(\eta, t)h^{(1)} = 0 \tag{28}$$

ε^2:

$$\sigma^2\phi^{(2)}_{\xi\eta} = -g^{(1)}h^{(1)} \tag{29}$$

ε^3:

$$g^{(3)}_{\xi t} + m_2(\xi, t)g^{(3)} + m_1(\eta, t)h^{(3)}$$
$$= -[D_\xi D_t g^{(1)} \cdot \phi^{(2)} + m_2(\xi, t)g^{(1)} \cdot \phi^{(2)} + m_1(\eta, t)h^{(1)} \cdot \phi^{(2)}]$$
$$h^{(3)}_{\eta t} + m_2(\xi, t)g^{(3)} + m_1(\eta, t)h^{(3)} \tag{30}$$
$$= -[D_\eta D_t h^{(1)} \cdot \phi^{(2)} + m_2(\xi, t)g^{(1)} \cdot \phi^{(2)} + m_1(\eta, t)h^{(1)} \cdot \phi^{(2)}]$$
$$h^{(1)}\phi^{(2)}_{\eta t} - g^{(1)}\phi^{(2)}_{\xi t} = 0$$

ε^4:

$$2\sigma^2\phi^{(4)}_{\xi\eta} + \sigma^2 D_\xi D_\eta \phi^{(2)} \cdot \phi^{(2)} = -2[h^{(1)}g^{(3)} + g^{(1)}h^{(3)}] \tag{31}$$

and so on.

(i) *Line kinks of sine–Gordon I* ($\sigma^2 = 1$). To generate soliton solutions of sine–Gordon I equation, one has to solve the variable coefficient equations of the type (28). To solve (28) explicitly, we make the following transformation by virtue of (9)):

$$g^{(1)} = \mathrm{i}u_\eta \qquad h^{(1)} = \mathrm{i}u_\xi \tag{32}$$

to convert (28) into

$$u_{\xi\eta t} + m_2(\xi, t)u_\eta + m_1(\eta, t)u_\xi = 0\,. \tag{33}$$

To solve the above variable coefficient equation, we look for the separation of variables

$$u(\xi, \eta, t) = P(\xi, t)Q(\eta, t) \tag{34}$$

to give rise to

$$Q_\eta[P_{\xi t} + m_2(\xi, t)P] + P_\xi[Q_{\eta t} + m_1(\eta, t)Q] = 0\,. \tag{35}$$

The above equation suggests that we should have

$$P_{\xi t} + m_2(\xi, t)P = kP_\xi \tag{36a}$$
$$Q_{\eta t} + m_1(\eta, t)Q = -kQ_\eta \tag{36b}$$

where k is a constant. Now, redefining $P \to \hat{P}\exp(kt)$, $Q \to \hat{Q}\exp(-kt)$ and dropping the hats, equation (36) becomes

$$P_{\xi t} + m_2(\xi, t)P = 0 \tag{37a}$$
$$Q_{\eta t} + m_1(\eta, t)Q = 0\,. \tag{37b}$$

To generate line kinks, we assume that the arbitrary functions $m_2(\xi, t)$ and $m_1(\eta, t)$ tend to constant values m_2 and m_1, respectively, as

$$m_2(\xi, t) \underset{|\xi|\to\infty}{\to} m_2,\ m_1(\eta, t) \underset{|\eta|\to\infty}{\to} m_1\,. \tag{38}$$

It should be mentioned that the asymptotic values of the boundaries should be non-zero in the case of (2+1)-dimensional sine–Gordon equation for non-trivial solutions to exist in contrast to Davey–Stewartson (DS) and Nizhnik–Novikov–Veselov (NNV) equations [4, 5, 7, 8] where one normally takes the asymptotic values of the boundaries to be zero for line soliton solutions. Hence, equation (37) becomes

$$P_{\xi t} + m_2 P = 0 \qquad Q_{\eta t} + m_1 Q = 0\,. \tag{39}$$

Equation (39) has solutions of the form

$$P = \sum_{i=1}^{N} \exp\left(2p_i\xi - \frac{m_2}{2p_i}t + c_i\right) \tag{40a}$$

$$Q = \sum_{i=1}^{N} \exp\left(2q_i\eta - \frac{m_1}{2q_i}t + c_i'\right) \tag{40b}$$

where p_i, q_i, c_i and c_i' are arbitrary constants so that

$$g^{(1)} = \mathrm{i}\sum_{i=1}^{N} 2q_i \exp(\chi_i) \qquad h^{(1)} = \mathrm{i}\sum_{i=1}^{N} 2p_i \exp(\chi_i)$$

$$\chi_i = 2p_i\xi + 2q_i\eta - \frac{m_1}{2q_i}t - \frac{m_2}{2p_i}t + \chi_i^{(0)} \qquad \chi_i^{(0)} : \text{are constants}\,. \tag{41}$$

To construct one line kink, we assume $N = 1$ and so we have

$$g^{(1)} = 2\mathrm{i}q_1 \exp(\chi_1) \qquad h^{(1)} = 2\mathrm{i}p_1 \exp(\chi_1)\,. \tag{42}$$

Now, equation (29) becomes

$$\phi^{(2)}_{\xi\eta} = 4p_1q_1 \exp(2\chi_1)\,. \tag{43}$$

Integrating (43), we get

$$\phi^{(2)} = \exp(2\chi_1 + 2\delta) \qquad \exp(2\delta) = \tfrac{1}{4}\,. \tag{44}$$

Hence, the one line kink solution of 2DSGI becomes

$$q = \frac{\varepsilon g^{(1)}}{1 + \varepsilon^2\phi^{(2)}} = \frac{2\mathrm{i}q_1 \exp(\chi_1)}{1 + \exp(2\chi_1 + 2\delta)} \tag{45a}$$

$$r = \frac{\varepsilon h^{(1)}}{1 + \varepsilon^2\phi^{(2)}} = \frac{2\mathrm{i}p_1 \exp(\chi_1)}{1 + \exp(2\chi_1 + 2\delta)}\,. \tag{45b}$$

Reverting back to the original field variable θ (cf equation (9)), one gets the familiar line kink solution as

$$\theta = 4\tan^{-1}[\exp(\chi_1 + \delta)]\,. \tag{46}$$

The above solution is identical to the one given by Konopelchenko [2]. The construction of multisoliton solutions is quite straightforward. One just takes any multisoliton solution in $g^{(1)}$ or $h^{(1)}$ and generates the corresponding elements in the truncated series (27) by solving (28)–(31).

(ii) *Line kinks for 2D sine–Gordon II*. In the case of sine–Gordon II equation ($\sigma^2 = -1$), the characteristic variables ξ and η are conjugate to each other as is evident from equation (5). As the field variable θ is always real, one has to impose certain constraints on the parameters as well as on the asymptotic values of the boundaries m_1 and m_2 to generate line kinks. They are

$$q_1 = p_1^* \qquad m_2 = m_1^* \qquad \xi = z, \eta = z^*\,. \tag{47}$$

Again, giving the transformation (by virtue of (9))

$$g^{(1)} = u_\eta \qquad h^{(1)} = u_\xi \qquad (\text{since } \sigma^2 = -1) \tag{48}$$

to equation (28), we end up with the equation (39). In view of the transformation (47), P and Q become conjugate to each other and hence their product becomes real. To generate one line kink, we now choose for $N = 1$ as

$$\begin{aligned} g^{(1)} &= 2p_1^* \exp(\chi_1') \qquad h^{(1)} = 2p_1 \exp(\chi_1') \\ \chi_1' &= 2p_1 z + 2p_1^* z^* - \frac{m_1}{2p_1^*}t - \frac{m_1^*}{2p_1}t + \chi_1'^{(0)}. \end{aligned} \tag{49}$$

Solving (29) for $\sigma^2 = -1$,

$$\phi^{(2)} = \exp(2\chi_1' + 2\delta') \qquad \exp(2\delta') = \tfrac{1}{4}\,. \tag{50}$$

Hence, the one line kink solution of 2D sine–Gordon II equation becomes

$$\theta = 4\tan^{-1}[\exp(\chi_1' + \delta')] \tag{51}$$

which is again in conformity with the one generated by Konopelchenko [2]. Multikink solutions can also be generated as in the case of sine–Gordon I equation by considering any multisoliton solution in $g^{(1)}$ or $h^{(1)}$ and generating the other elements of the corresponding truncated series by solving the remaining equations.

5. Localized coherent structures for 2D sine–Gordon I (2DSGI) equation ($\sigma^2 = 1$)

We shall now bring out the significance of the boundaries $m_2(\xi, t)$ and $m_1(\eta, t)$ and thereby invoke the concept of 'ghost solitons' driving the boundaries before generating localized solutions of 2DSGI equation. Now, from equations (4) and (6), the two arbitrary potentials ρ_ξ and ρ_η for the 2DSGI equation can be expressed as

$$\tfrac{1}{2}\rho_\xi = m_2(\xi, t) + \tfrac{1}{4}\int_{-\infty}^{\eta} \mathrm{d}\eta'(\theta_\xi\theta_{\eta'})_t \tag{52a}$$

$$\tfrac{1}{2}\rho_\eta = m_1(\eta, t) + \tfrac{1}{4}\int_{-\infty}^{\xi} \mathrm{d}\xi'(\theta_\eta\theta_{\xi'})_t \tag{52b}$$

where $m_1(\eta, t)$ and $m_2(\xi, t)$ are the two non-trivial boundaries. From the above equations, it is evident that even if the physical field variable θ vanishes (correspondingly q or r in (9) vanish), the potentials ρ_ξ and ρ_η are driven by the two boundaries $m_2(\xi, t)$ and $m_1(\eta, t)$, respectively. Thus, one can indeed invoke the familiar concept of ghost solitons in 2DSGI equation also similar to DS and NNV equations [16, 8]. To generate the ghost solitons, which drive the boundaries for constructing localized solutions, one has to solve the variable coefficient differential equations (37) as such.

To generate localized solutions, we now choose

$$g^{(1)} = \mathrm{i}u_\eta = \mathrm{i}\zeta P(\xi, t)Q_\eta(\eta, t) \tag{53}$$

$$h^{(1)} = \mathrm{i}u_\xi = \mathrm{i}\zeta P_\xi(\xi, t)Q(\eta, t) \tag{54}$$

where $P(\xi, t)$ and $Q(\eta, t)$ are the solutions of (37) and ζ is a constant parameter. Then, equation (29) in the case of sine–Gordon I takes the form

$$\phi^{(2)}_{\xi\eta} = \zeta^2(PP_\xi)(QQ_\eta)\,. \tag{55}$$

Solving equation (55), we have

$$\phi^{(2)} = \frac{\zeta^2}{4}P^2Q^2\,. \tag{56}$$

To construct exponentially localized dromion solutions, one has to solve equation (37) explicitly. As the boundaries have non-zero asymptotic values for non-trivial solutions, one can rewrite equation (37) as

$$P_{\xi t} + [m_2'(\xi, t) + m_2]P = 0 \tag{57a}$$
$$Q_{\eta t} + [m_1'(\eta, t) + m_1]Q = 0 \tag{57b}$$

where we have expressed the functions $m_1(\eta, t)$ and $m_2(\xi, t)$ as

$$m_1(\eta, t) = m_1 + m_1'(\eta, t) \tag{58a}$$
$$m_2(\xi, t) = m_2 + m_2'(\xi, t) \qquad m_1, m_2 \text{ are constants}. \tag{58b}$$

As we expect the boundaries to be driven by ghost solitons (wavelike solutions) for localized structures, we now assume them to have the specific form

$$m_2'(\xi, t) = m_2'(\xi + V_2 t) = m_2'(\xi') \tag{59a}$$
$$m_1'(\eta, t) = m_1'(\eta + V_1 t) = m_1'(\eta'). \tag{59b}$$

Then, equation (57) is now reduced to the stationary, time-independent Schrödinger equation as

$$P_{\xi'\xi'} + \left[\frac{1}{V_2} m_2'(\xi') + \frac{m_2}{V_2}\right] P = 0 \tag{60a}$$
$$Q_{\eta'\eta'} + \left[\frac{1}{V_1} m_1'(\eta') + \frac{m_1}{V_1}\right] Q = 0. \tag{60b}$$

Redefining $m_2/V_2 = k^2$ and $m_1/V_1 = \omega^2$, we have

$$P_{\xi'\xi'} + \left[\frac{1}{V_2} m_2'(\xi') + k^2\right] P = 0 \tag{61a}$$
$$Q_{\eta'\eta'} + \left[\frac{1}{V_1} m_1'(\eta') + \omega^2\right] Q = 0. \tag{61b}$$

Let $k_j = \mathrm{i}p_j$, $\omega_k = \mathrm{i}q_k$, p_j, q_k and P_j, Q_k, $1 \leqslant j \leqslant L$, $1 \leqslant k \leqslant M$ be the discrete eigenvalues and eigenfunctions associated with $m_2'(\xi')$ and $m_1'(\eta')$, respectively. If the reflection coefficients associated with the potentials $m_2'(\xi')$ and $m_1'(\eta')$ are zero, the discrete eigenfunctions can be found in closed form as [17]

$$P_n + \sum_{j=1}^{L} \frac{a_n a_j}{p_n + p_j} \exp[-(p_n + p_j)\xi_j']P_j = a_n \exp(-p_n \xi') \tag{62a}$$
$$Q_n + \sum_{k=1}^{M} \frac{b_n b_k}{q_n + q_k} \exp[-(q_n + q_k)\eta_k']Q_k = b_n \exp(-q_n \eta') \tag{62b}$$

with the potentials being given by

$$m_2' = -2V_2 \sum_{j=1}^{L} a_n[(\exp\{-p_j \xi'\}) P_j]_{\xi'} \tag{63a}$$
$$m_1' = -2V_1 \sum_{k=1}^{M} b_n[(\exp\{-q_k \eta'\}) Q_k]_{\eta'}. \tag{63b}$$

To generate the (1, 1) dromion solution, we take $L = M = 1$ and so we have

$$P_1 = \frac{a_1 \exp\{-p_1\xi'\}}{1+\exp\{-2(p_1\xi' - \delta_1)\}} \qquad \exp(2\delta_1) = \frac{a_1^2}{2p_1} \tag{64a}$$

$$Q_1 = \frac{b_1 \exp\{-q_1\eta'\}}{1+\exp\{-2(q_1\eta' - \delta_2)\}} \qquad \exp(2\delta_2) = \frac{b_1^2}{2q_1} \tag{64b}$$

$$m_2' = 2V_2 p_1^2 \operatorname{sech}^2(p_1\xi' - \delta_1) \tag{64c}$$

$$m_1' = 2V_1 q_1^2 \operatorname{sech}^2(q_1\eta' - \delta_2)\,. \tag{64d}$$

Using equation (64), we now choose in accordance with (56)

$$\phi = 1 + \frac{\zeta^2}{4}\frac{a_1^2 b_1^2 \exp\{-2(p_1\xi' + q_1\eta')\}}{(1+\exp\{-2(p_1\xi' - \delta_1)\})^2(1+\exp\{-2(q_1\eta' - \delta_2)\})^2} \tag{65}$$

and substitute this in equation (26*c*) ($\sigma^2 = 1$), we obtain

$$\frac{\zeta^2 a_1^2 b_1^2 p_1 q_1 \exp\{-2(p_1\xi' + q_1\eta')\}[\exp\{-2(p_1\xi' - \delta_1)\} - 1][\exp\{-2(q_1\eta' - \delta_2)\} - 1]}{(1+\exp\{-2(p_1\xi' - \delta_1)\})^3(1+\exp\{-2(q_1\eta' - \delta_2)\})^3} = -gh \tag{66}$$

where ζ is some constant parameter. This suggests that the functions g and h should have the form in accordance with (53) and (54) as

$$g = \mathrm{i}\zeta\left[\frac{a_1\exp(-p_1\xi')}{(1+\exp\{-2(p_1\xi' - \delta_1)\})}\right]\left[\frac{b_1 q_1 \exp(-q_1\eta')[\exp\{-2(q_1\eta' - \delta_2)\} - 1]}{(1+\exp\{-2(q_1\eta' - \delta_2)\})^2}\right] \tag{67a}$$

$$h = \mathrm{i}\zeta\left[\frac{a_1 p_1 \exp(-p_1\xi')[\exp\{-2(p_1\xi' - \delta_1)\} - 1]}{(1+\exp\{-2(p_1\xi' - \delta_1)\})^2}\right]\left[\frac{b_1\exp(-q_1\eta')}{(1+\exp\{-2(q_1\eta' - \delta_2)\})}\right]. \tag{67b}$$

Substituting (65) and (67) in (25), one finally gets the (1,1) dromion solution through (9) as

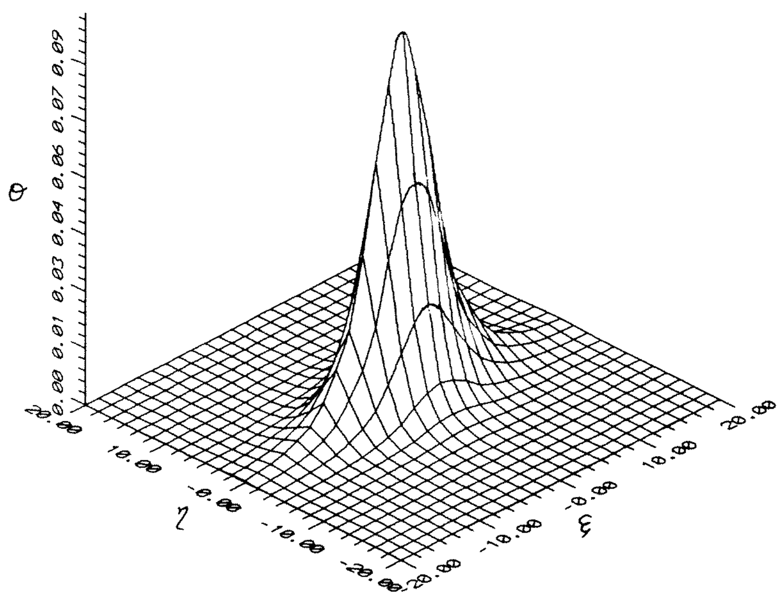

Figure 1. Localized solution of two-dimensional sine–Gordon I equation at $t = 0.1$ with the parameters $p_1 = 0.3$, $q_1 = 0.6$, $V_1 = -0.2$, $V_2 = -0.4$, $\zeta = 0.1$.

$$\theta = 4\tan^{-1}\left[\frac{\zeta\,\exp\{-[p_1\xi' + q_1\eta']\}}{(1+\exp\{-2(p_1\xi' - \delta_1)\})(1+\exp\{-2(q_1\eta' - \delta_2)\})}\right] \tag{68}$$

which is plotted in figure 1. The above solution is concurrent with the one given by Dubrovsky and Konopelchenko [13] who obtained it through the inverse scattering transform procedure in the d-bar formalism. In our analysis, we have reduced the equation (37) into an algebraic equation (62) so that one can indeed generalize it to multidromions by considering any number of bound states even though the actual analysis proves to be cumbersome and unmanageable by hand calculation.

In the case of 2D sine–Gordon II equation, as we have noted earlier, the conjugate nature of the independent variables imposes constraints on the parameters as well as on the boundaries $m_1(\eta, t)$ and $m_2(\xi, t)$ which are now complex. Hence, it is not clear what will be the nature of the spectrum of solutions of (37) subject to (47). Thus, the existence of localized structures of 2DSGII equation essentially depends on the solvability of the equation (37) subject to the constraints (47) and this remains an open question.

6. Discussion

In this paper, we have carried out the singularity structure analysis of the 2DSG equation and shown that it admits the Painlevé (P) property. We have then derived its bilinear form with variable coefficients straightforwardly from the P-analysis and then used it to generate line kinks for both the 2D sine–Gordon I and II equations by treating the boundaries as non-zero constants. We have then generated localized solutions of the sine–Gordon I equation by driving the boundaries through two non-parallel ghost solitons. We have also brought out the significance of the non-zero asymptotic values of the boundaries in sharp contrast to the DS and NNV equations. Existence of localized solutions of 2DSGII equation remains an open question depending upon the solvability of equation (37) subject to the constraints (47).

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